An Illustration of Kolchin's Proof of [Kolchin 1973, Prop. 10, page 200]

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In [1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

In [2]:
```
x = var ('x')
y3, y2, y1, rho, alpha, phi, c = function ('y3, y2, y1, rho, alpha, phi, c')
```

In [3]:
```
R = DifferentialRing (derivations = [x], blocks = [c, y3, y2, y1, rho, alpha, phi])
```

The base field F contains some phi which is not a constant

In [4]:
```
phi_defining_equation = Derivative(phi(x),x,x) - 1
phi_defining_equation
```

Out[4]:

$$\frac{d^2}{dx^2}\phi(x) - 1$$

The characteristic set A of the prime ideal p0 of F[y1,y2,y3]

In [5]:
```
A = (y3(x) - y2(x))**2 - Derivative(phi(x),x)*y1(x)**3
A
```

Out[5]:

$$\left(-\mathrm{y}_2(x) + \mathrm{y}_3(x)\right)^2 - \mathrm{y}_1{}^3(x)\frac{d}{dx}\phi(x)$$

In [6]:
```
H_A = R.separant (A)
H_A
```

Out[6]: $-2\,\mathrm{y}_2\,(x) + 2\,\mathrm{y}_3\,(x)$

alpha is differentially algebraic over F and permits to build a zero Alpha = (0, alpha, alpha) of A (G = F)

A singular zero is chosen (it annihilates H_A)

In [7]:
```
alpha_defining_equation = Derivative(alpha(x),x)**2 - phi(x)*alpha(x)
alpha_defining_equation
```

Out[7]:

$$-\alpha(x)\phi(x) + \left(\frac{d}{dx}\alpha(x)\right)^2$$

In [8]:
```
Alpha = { y1(x):0, y2(x):alpha(x), y3(x):alpha(x) }
Alpha
```

Out[8]: $\{\mathrm{y}_1\,(x) : 0,\ \ \mathrm{y}_2\,(x) : \alpha(x),\ \ \mathrm{y}_3\,(x) : \alpha(x)\}$

In [9]:
```
R.evaluate (A, Alpha)
```

Out[9]: 0

In [10]:
```
R.evaluate (H_A, Alpha)
```

Out[10]: 0

Beta is a (rational parametrization of a) Puiseux series in c

1. centered at Alpha (Beta(0) = Alpha)
2. it annihilates A
3. it does not annihilate H_A
4. requires a (differential) algebraic extension (rho) of G = F (L = G)

In [11]:
```
rho_defining_equation = rho(x)**2 - Derivative(phi(x),x)
rho_defining_equation
```

Out[11]:

$$\rho^2(x) - \frac{d}{dx}\phi(x)$$

In [12]:
```
Beta = { y1(x):c(x)**2, y2(x):alpha(x), y3(x):alpha(x) + rho(x)*c(x)**3 }
Beta
```

Out[12]:

$$\{\mathrm{y}_1(x) : c^2(x),\ \mathrm{y}_2(x) : \alpha(x),\ \mathrm{y}_3(x) : \alpha(x) + c^3(x)\rho(x)\}$$

In [13]:
```
R.evaluate (A, Beta)
```

Out[13]:

$$c^6(x)\rho^2(x) - c^6(x)\frac{d}{dx}\phi(x)$$

In [14]:
```
rem (R.evaluate (A, Beta), rho_defining_equation, rho(x))
```

Out[14]: 0

In [15]:
```
R.evaluate (H_A, Beta)
```

Out[15]:

$$2c^3(x)\rho(x)$$

The last steps of the proof. Pick a differential polynomial f in [A]:H_A^\infty

Then f(Beta) is differential power series in L{{c}} which must be zero

In [16]:
```
f = Derivative(A,x,x) + y1(x)*Derivative(A,x)
f
```

Out[16]:

$$\mathrm{y}_1(x)\frac{d}{dx}\left((-\mathrm{y}_2(x) + \mathrm{y}_3(x))^2 - \mathrm{y}_1{}^3(x)\frac{d}{dx}\phi(x)\right) + \frac{d^2}{dx^2}\left((-\mathrm{y}_2(x) + \mathrm{y}_3(x))^2 - \mathrm{y}_1{}^3(x)\frac{d}{dx}\phi(x)\right)$$

In [17]:
```
series = R.evaluate (f.doit(), Beta).doit ()
```

In [18]:
```
koeffs, terms = R.coeffs(series, c(x))
terms
```

Out[18]:

$$\left[c^5(x)\frac{d^2}{dx^2}c(x),\ c^4(x)\left(\frac{d}{dx}c(x)\right)^2,\ c^7(x)\frac{d}{dx}c(x),\ c^5(x)\frac{d}{dx}c(x),\ c^8(x),\ c^6(x)\right]$$

In [19]:
```
koeffs
```

Out[19]:

$$\left[6\rho^2(x) - 6\frac{d}{dx}\phi(x),\ 30\rho^2(x) - 30\frac{d}{dx}\phi(x),\ 6\rho^2(x) - 6\frac{d}{dx}\phi(x),\ 24\rho(x)\frac{d}{dx}\rho(x) - 12\frac{d^2}{dx^2}\phi(x),\ 2\rho(x)\frac{d}{dx}\rho(x) - \frac{d^2}{dx^2}\phi(x),\ 2\rho(x)\frac{d}{dx}\right.$$

Computing in L or in L{{c}} amounts to taking normal forms of expressions modulo the characteristic set C defining our successive field extensions

In [20]:
```
C = RegularDifferentialChain ([rho_defining_equation,
         alpha_defining_equation, phi_defining_equation], R)
C.equations(solved=True)
```

Out[20]:

$$\left[\frac{d^2}{dx^2}\phi(x) = 1,\ \left(\frac{d}{dx}\alpha(x)\right)^2 = \alpha(x)\phi(x),\ \rho^2(x) = \frac{d}{dx}\phi(x)\right]$$

In [21]:
```
C.normal_form (koeffs)
```

Out[21]: $[0,\ 0,\ 0,\ 0,\ 0,\ 0]$

Last, ``a diagram commutes'' but I have not found any convincing way to illustrate this subtle step by a computation

In [ ]: